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Solutions of the q-deformed Schrödinger equation for special potentials

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Abstract

Solutions of the q-deformed Schrödinger equation are presented for the following potentials: shifted oscillator, isotropic oscillator, Rosen–Morse II, Eckart II, and Poschl–Teller I and II potentials. Various properties of solutions to such equations are discussed including the limit case $q \to 1$ that corresponds to the non-deformed Schrödinger equation.

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1. Introduction

In this paper, we study the solution of the eigenvalue problem

$$-\partial_q^2\psi(x) + W(x)\partial_q + V(x)\psi(x) = E\psi(x), \tag{1.1}$$

where $0 < q < 1$ and

$$\partial_q\psi(x) := \frac{\psi(x) - \psi(qx)}{(1-q)x}, \tag{1.2}$$

for the family of the second-order q-difference operators which include q-Schrödinger operators with potentials being a q-deformation of the shifted oscillator, isotropic oscillator, Rosen–Morse II, Eckart II, and Poschl–Teller I and II potentials.

Our investigation will be based on the factorization method [12] as well as on the theory of the classical q-orthogonal polynomials related to q-difference Hahn equation [11, 13]. In the limit $q \to 1$, the q-difference equation (1.1) becomes the second-order differential equation which could be interpreted as one-dimensional Schrödinger equation or the radial part of the three-dimensional Schrödinger equation. Another motivation for the investigation of the problem in question is related to the possibility to use the q-difference equation (1.1) as an intermediate step for the numerical treatment of the corresponding differential equation. There is also some mathematical reason, since spectral analysis of the second-order q-difference operators gives a link between the theory of q-special functions [8] and spectral analysis of Jacobi-type operators [16].

The content of the paper is following. In section 2, we construct the chain of Hilbert spaces $\mathcal{H}_k$, $k \in \mathbb{N} \cup \{0\}$, of functions which are square integrable with respect to the Jackson q-integral with a weight function. It is also shown that in the limit $q \to 1$ one obtains Hilbert spaces related to the classical orthogonal polynomials as well as to the eigenvalue problems for the Schrödinger operators.

The family of solutions of the eigenvalue problem of the corresponding q-difference Schrödinger operators $\mathbf{H}_k : \mathcal{H}_k \to \mathcal{H}_k$ with various potentials is presented in section 3. All these solutions are expressed in terms of the basic hypergeometric series.

2. A chain of the factorized q-difference operators

Let us consider the sequence of the vector spaces $\mathcal{V}_k$, $k \in \mathbb{N} \cup \{0\}$, consisting of the complex-valued functions $\psi : [a, b]_q \to \mathbb{C}$ on the q-interval

$$[a, b]_q := \{q^n a : n \in \mathbb{N} \cup \{0\}\} \cup \{q^n b : n \in \mathbb{N} \cup \{0\}\}. \tag{2.1}$$

We also assume that $\mathcal{V}_k$ are equipped with the scalar products

$$\langle \psi | \varphi \rangle_k := \int_a^b \overline{\psi(x)} \varphi(x) \varrho_k(x) \, \mathrm{d}_q x \tag{2.2}$$

defined by Jackson q-integral [8]

$$\int_a^b \psi(x) \, \mathrm{d}_q x := \sum_{n=0}^{\infty} (1 - q) q^n b \psi(q^n b) - \sum_{n=0}^{\infty} (1 - q) q^n a \psi(q^n a) \tag{2.3}$$

taken over $[a, b]_q$ and dependent on the weight function $\varrho_k : [a, b]_q \to \mathbb{R}$, where $0 < q < 1$.

In the case when $a = 0$ and $b = \infty$, by the definition one assumes

$$\int_0^{\infty} \psi(x) \, \mathrm{d}_q x := \lim_{n \to \infty} \int_0^{q^{-n}} \psi(x) \, \mathrm{d}_q x = \sum_{n=-\infty}^{\infty} (1 - q) q^n f(q^n) \tag{2.4}$$

and in the case if $a = -\infty$ and $b = \infty$ we define

$$\int_{-\infty}^{\infty} \psi(x) \, \mathrm{d}_q x := \lim_{n \to \infty} \int_{-q^{-n}}^{q^{-n}} \psi(x) \, \mathrm{d}_q x = \sum_{n=-\infty}^{\infty} (1 - q) q^n \psi(q^n) + \sum_{n=-\infty}^{\infty} (1 - q) q^n \psi(-q^n), \tag{2.5}$$

see [8]. Let us remark here that the scalar products (2.2) are not positively defined in the general case.

The main object of our considerations will be the sequence

$$\mathbf{H}_k = Z_k(x) \partial_q Q^{-1} \partial_q + W_k(x) \partial_q + V_k(x), \tag{2.6}$$

$k \in \mathbb{N} \cup \{0\}$, of the q-difference operators $\mathbf{H}_k : \mathcal{V}_k \to \mathcal{V}_k$ symmetric

$$\langle \psi | \mathbf{H}_k \varphi \rangle_k = \langle \mathbf{H}_k \psi | \varphi \rangle_k \tag{2.7}$$

with respect to the scalar products (2.2), where $\psi, \varphi \in \mathcal{D}(\mathbf{H}_k)$. The definition of the domain $\mathcal{D}(\mathbf{H}_k)$ for $\mathbf{H}_k$ will be given below. Condition (2.7) gives the following relationships:

$$\partial_q (Z_k \varrho_k) = W_k \varrho_k, \tag{2.8}$$

$$Z_k \varrho_k (\bar{\psi} Q^{-1} \partial_q \varphi - \varphi Q^{-1} \partial_q \bar{\psi})|_a^b = 0 \tag{2.9}$$

for the functions Z_k, W_k and ϱ_k, where (2.9) holds for any ψ and φ from $\mathcal{D}(\mathbf{H}_k)$. Let us note here that $q[a,b]_q \subseteq [a,b]_q$. Hence, the q-difference operators

$$Q\psi(x) := \psi(qx), \tag{2.10}$$

$$\partial_q \psi(x) := \frac{\psi(x) - \psi(qx)}{(1-q)x}, \tag{2.11}$$

and thus the q-difference operator (2.6), are correctly defined in $\mathcal{V}_k$.

Assuming in (2.6) that

$$Z_k(x) = -\frac{q^{-2k\gamma+\gamma-1}}{[\gamma]^2} x^{2(1-\gamma)} B_k(x)(1 + (1-q^\gamma)q^{k\gamma-\gamma} x^\gamma f_k(q^{-1}x)), \tag{2.12}$$

$$W_k(x) = \frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} \left(B_k(x) \left(f_k(x) - q^{-1} f_k(q^{-1}x) - \frac{[1-\gamma]}{[\gamma]} q^{-k\gamma+\gamma-1} x^{-\gamma} \right) \right.$$
$$\left. - A_k(1 + (1-q^\gamma)q^{k\gamma} x^\gamma f_k(x)) \right), \tag{2.13}$$

$$V_k(x) = -\frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} B_k(x) \partial_q Q^{-1} f_k(x) - A_k(x) f_k(x)(1 + (1-q^\gamma)q^{k\gamma} x^\gamma f_k(x))$$
$$+ B_k(x) f_k^2(x) + a_k, \tag{2.14}$$

we factorize

$$\mathbf{H}_k = \mathbf{A}_k{}^* \mathbf{A}_k + a_k, \qquad a_k \in \mathbb{R}, \tag{2.15}$$

the second-order operators $\mathbf{H}_k$ as a product of two first-order q-difference operators $\mathbf{A}_k : \mathcal{V}_k \to \mathcal{V}_{k-1}$ and $\mathbf{A}_k^* : \mathcal{V}_{k-1} \to \mathcal{V}_k$ given by

$$\mathbf{A}_k = \frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} \partial_q + f_k, \tag{2.16}$$

$$\mathbf{A}_k^* = \left(\frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} \partial_q + f_k \right)^*$$
$$= -\frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} B_k \partial_q Q^{-1} + B_k f_k - A_k(1 + (1-q^\gamma)q^{k\gamma} x^\gamma f_k), \tag{2.17}$$

where as usual $[\gamma] := \frac{1-q^\gamma}{1-q}$, see [6, 7, 9, 10]. In the following, according to [6], we will assume that

$$B_k(x) = q^{2k\gamma-k} x^{2(\gamma-1)} B(q^k x), \tag{2.18}$$

$$A_k(x) = \frac{q^{k\gamma-k}}{1-q^\gamma} x^{\gamma-2} (B(q^k x) - q^{2k(1-\gamma)} B(x)), \tag{2.19}$$

$$f_k(x) = \frac{q^{-k+\frac{\gamma-1}{2}}}{(1-q^\gamma)x^\gamma} \sqrt{\frac{D(qx)}{B(x)}} - \frac{1}{(1-q^\gamma)q^{k\gamma} x^\gamma}, \tag{2.20}$$

where

$$B(x) = b_2 x^2 + b_1 x + b_0, \tag{2.21}$$

$$D(x) = (b_2 + (1-q^\gamma)[\gamma] q^{-\gamma}(a_0 - a_1))x^2 + (b_1 + (1-q^\gamma)c)x + b_0, \tag{2.22}$$

$b_0, b_1, b_2, c, a_0, a_1 \in \mathbb{R}$ and $\gamma > 1$. We require that the parameters a_0, a_1, c, b_0 do not depend on q.

Now, let us fix a solution ψ_k^0 of the q-difference equation

$$\mathbf{A}_k \psi_k^0 = 0 \tag{2.23}$$

and choose $\varrho_k : [a, b]_q \to \mathbb{R}$ in such a way which ensure the positivity of the scalar product

$$\langle \psi_k | \psi_k \rangle_k := \int_a^b |P_k(x)|^2 \left|\psi_k^0(x)\right|^2 \varrho_k(x)\, \mathrm{d}_q x, \tag{2.24}$$

for $\psi_k = P_k \psi_k^0 \in \mathcal{V}_k$. Then, we define the unitary space

$$\mathcal{H}_k := \left\{ P_k \psi_k^0 \in \mathcal{V}_k : \langle \psi_k | \psi_k \rangle_k < +\infty \right\}, \tag{2.25}$$

which in special case could be Hilbert space. If it is not the case, we complete $\mathcal{H}_k$ to be the Hilbert space by the standard completion procedure. Restricting the operator $\mathbf{H}_k : \mathcal{H}_k \to \mathcal{H}_k$ (2.6) to the space $\mathcal{H}_k$ and keeping in the mind the factorization conditions (2.12), (2.13) and (2.14) we find that the symmetricity conditions (2.8) and (2.9) for $\mathbf{H}_k$ take the following form:

$$\partial_q \left(\frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} B_k \varrho_k \right) = A_k \varrho_k, \tag{2.26}$$

$$x^{1-2\gamma} B_k \varrho_k \left|\psi_k^0\right|^2 (\bar{P}_k Q^{-1} R_k - R_k Q^{-1} \bar{P}_k)\Big|_a^b = 0, \tag{2.27}$$

for $P_k \psi_k^0, R_k \psi_k^0 \in \mathcal{D}(\mathbf{H}_k) \subset \mathcal{H}_k$.

In order to avoid such restrictive condition on the domain $\mathcal{D}(\mathbf{H}_k)$, we replace (2.27) by the stronger condition

$$x^{1-2\gamma} B_k \varrho_k \left|\psi_k^0\right|^2 \Big|_a^b = 0, \tag{2.28}$$

which one can consider as a boundary condition for the q-Person equation (2.26).

In the limit $q \to 1$, since $\partial_q \to \frac{\mathrm{d}}{\mathrm{d}x}$, the operator (2.15) tends to the second-order ordinary differential operator

$$\mathbf{H}_k^1 = \mathbf{A}_k^{1*} \mathbf{A}_k^1 + a_k^1, \tag{2.29}$$

with the operators $\mathbf{A}_k^1$ and $\mathbf{A}_k^{1*}$ given by

$$\mathbf{A}_k^1 = \gamma^{-1} x^{1-\gamma} \frac{\mathrm{d}}{\mathrm{d}x} + \frac{2k(\gamma^{-1} - 1) - 1}{2x^\gamma} + \frac{\left(\gamma a_0 - \gamma a_1 - \gamma^{-1} b_2^1\right) x^2 + cx + \gamma^{-1} b_0}{2x^\gamma B^1(x)}, \tag{2.30}$$

$$\mathbf{A}_k^{1*} = -\gamma^{-1} x^{\gamma-1} B^1(x) \frac{\mathrm{d}}{\mathrm{d}x} + k\gamma^{-1} x^{\gamma-1} \left(\frac{\mathrm{d}}{\mathrm{d}x} B^1(x) \right) + \frac{(2k(1 - \gamma^{-1}) - 1) B^1(x) + \left(\gamma a_0 - \gamma a_1 - \gamma^{-1} b_2^1\right) x^2 + cx + \gamma^{-1} b_0}{2x^{2-\gamma}}. \tag{2.31}$$

These operators act in the Hilbert space $\mathcal{H}_k^1$ which consists of the complex-valued functions square integrable with respect to the scalar product

$$\left\langle \psi_k^1 \middle| \psi_k^1 \right\rangle_k := \int_a^b \left|P_k^1(x)\right|^2 \left|{}^1\psi_k^0(x)\right|^2 \varrho_k^1(x)\, \mathrm{d}x, \tag{2.32}$$

obtained from (2.24) when $q \to 1$. Let us note here that the set $[a, b]_q$ becomes the usual interval which we denote by $[a, b]_1$ and the q-integrals (2.3), (2.4) and (2.5) converge to the corresponding standard integrals in the limit $q \to 1$ and $\varrho_k^1 = \lim_{q\to 1} \varrho_k$. The parameters b_2^1 and b_1^1 appearing in (2.30) and (2.31) are defined by

$$B^1(x) := b_2^1 x^2 + b_1^1 x + b_0 = \lim_{q\to 1} B(x). \tag{2.33}$$

In the limit case, when $q \to 1$, we use the following notation:

$$\psi_k^1(x) = P_k^1(x)^1\psi_k^0(x), \tag{2.34}$$

where

$$P_k^1(x) = \lim_{q\to 1} P_k(x), \tag{2.35}$$

$${}^1\psi_k^0(x) = \lim_{q\to 1} \psi_k^0(x). \tag{2.36}$$

The solution of (2.23) takes the following form:

$${}^1\psi_k^0(x) = Cx^{k(\gamma-1)+\frac{\gamma}{2}} \exp\left(-\int \frac{(\gamma^2 a_0 - \gamma^2 a_1 - b_2^1)x^2 + \gamma c x + b_0}{2xB^1(x)}\,\mathrm{d}x\right), \tag{2.37}$$

when $q \to 1$. The formula

$$\frac{\left(\frac{x}{x_i}; q\right)_\infty}{\left(\frac{qx}{y_i}; q\right)_\infty} = \left(1 - \frac{x}{x_i^1}\right)^{-\frac{1}{x_i^1}\lim_{q\to 1}\frac{qx_i - y_i}{1-q}}, \tag{2.38}$$

where

$$(a; q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a), \tag{2.39}$$

$k \in \mathbb{N} \cup \{\infty\}$ and $i = 1, 2$, will also be useful in the intermediate calculations.

Having defined B_k, A_k and f_k by (2.18), (2.19) and (2.20) one obtains the families of solutions of the q-Pearson equation (2.26) and equation (2.23). They are specified by parameters $b_0, b_1, b_2, c, a_0, a_1 \in \mathbb{R}$. In formulae given below we denote the roots of the polynomial B by $x_1 \neq 0$, $x_2 \neq 0$. Similarly, by $y_1 \neq 0$, $y_2 \neq 0$ we denote the roots of the polynomial D. According to this notation one has the following list of the admissible scalar products:

(i) If $b_2 \neq 0$, $b_0 \neq 0$ and $b_2 + (1-q^\gamma)[\gamma]q^{-\gamma}(a_0 - a_1) \neq 0$, then

$$\varrho_k(x) = \frac{x^{(1-\gamma)(2k+1)}}{\left(\frac{x}{x_1}; q\right)_{k+1}\left(\frac{x}{x_2}; q\right)_{k+1}}, \tag{2.40}$$

$$\varrho_k^1(x) = \frac{x^{(1-\gamma)(2k+1)}}{(x-x_1)^{k+1}(x-x_2)^{k+1}}, \tag{2.41}$$

$$[a, b]_q = [q^{-k}x_1, q^{-k}x_2]_q, \qquad x_1 < x_2, \tag{2.42}$$

$$[a, b]_1 = [x_1, x_2], \tag{2.43}$$

$$\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}\sqrt{\frac{\left(\frac{x}{x_1}; q\right)_\infty\left(\frac{x}{x_2}; q\right)_\infty}{\left(\frac{qx}{y_1}; q\right)_\infty\left(\frac{qx}{y_2}; q\right)_\infty}}, \tag{2.44}$$

$$\begin{aligned}{}^1\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}(x-x_1)^{\frac{2b_2-\gamma^2(a_0-a_1)}{4b_2} - \frac{\frac{\gamma^2 b_1(a_0-a_1)}{2b_2} - \gamma c}{2\sqrt{b_1^2-4b_2b_0}}}\\ \times (x-x_2)^{\frac{2b_2-\gamma^2(a_0-a_1)}{4b_2} - \frac{\frac{\gamma^2 b_1(a_0-a_1)}{2b_2} - \gamma c}{2\sqrt{b_1^2-4b_2b_0}}}.\end{aligned} \tag{2.45}$$

(ii) If $b_2 \neq 0, b_0 \neq 0, b_2 + (1-q^\gamma)[\gamma]q^{-\gamma}(a_0 - a_1) = 0$ and $b_1 + (1-q^\gamma)c \neq 0$, then

$$\varrho_k(x) = \frac{x^{(1-\gamma)(2k+1)}}{\left(\frac{x}{x_1};q\right)_{k+1}\left(\frac{x}{x_2};q\right)_{k+1}}, \tag{2.46}$$

$$\varrho_k^1(x) = \frac{x^{(1-\gamma)(2k+1)}}{\left(x+\frac{b_0}{b_1}\right)^{k+1}}, \tag{2.47}$$

$$[a,b]_q = [q^{-k}x_1, q^{-k}x_2]_q, \tag{2.48}$$

$$[a,b]_1 = \left[-\frac{b_0}{b_1}, \infty\right], \tag{2.49}$$

$$\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}\sqrt{\frac{\left(\frac{x}{x_1};q\right)_\infty\left(\frac{x}{x_2};q\right)_\infty}{\left(-\left(\frac{b_1}{b_0}+(1-q^\gamma)\frac{c}{b_0}\right)qx;q\right)_\infty}}, \tag{2.50}$$

$${}^1\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}\left(x+\frac{b_0}{b_1}\right)^{\frac{b_1 - c\gamma + \gamma^2(a_0-a_1)\frac{b_0}{b_1}}{2b_1}} \exp\left(-\frac{\gamma^2(a_0-a_1)}{2b_1}x\right). \tag{2.51}$$

(iii) If $b_2 \neq 0, b_0 \neq 0, b_2 + (1-q^\gamma)[\gamma]q^{-\gamma}(a_0 - a_1) = 0$ and $b_1 + (1-q^\gamma)c = 0$, then

$$\varrho_k(x) = \frac{x^{(1-\gamma)(2k+1)}}{\left(\frac{x}{x_1};q\right)_{k+1}\left(\frac{x}{x_2};q\right)_{k+1}}, \tag{2.52}$$

$$\varrho_k^1(x) = x^{(1-\gamma)(2k+1)}, \tag{2.53}$$

$$[a,b]_q = [q^{-k}x_1, q^{-k}x_2]_q, \tag{2.54}$$

$$[a,b]_1 = [-\infty, \infty], \tag{2.55}$$

$$\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}\sqrt{\left(\frac{x}{x_1};q\right)_\infty\left(\frac{x}{x_2};q\right)_\infty}, \tag{2.56}$$

$${}^1\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}\exp\left(-\frac{\gamma^2(a_0-a_1)}{4b_0}x^2 - \frac{\gamma c}{2b_0}x\right). \tag{2.57}$$

(iv) If $b_2 \neq 0, b_1 \neq 0, b_0 = 0, b_2 + (1-q^\gamma)[\gamma]q^{-\gamma}(a_0 - a_1) \neq 0$ and $b_1 + (1-q^\gamma)c \neq 0$, then

$$\varrho_k(x) = \frac{x^{k(1-2\gamma)-\gamma}}{\left(-\frac{b_2}{b_1}x;q\right)_{k+1}}, \tag{2.58}$$

$$\varrho_k^1(x) = \frac{x^{k(1-2\gamma)-\gamma}}{\left(x+\frac{b_1}{b_2}\right)^{k+1}}, \tag{2.59}$$

$$[a,b]_q = \left[0, -q^{-k}\frac{b_1}{b_2}\right]_q \quad \text{for} \quad \frac{b_1}{b_2} < 0 \quad \text{or}$$
$$[a,b]_q = \left[-q^{-k}\frac{b_1}{b_2}, 0\right]_q \quad \text{for} \quad \frac{b_1}{b_2} > 0, \tag{2.60}$$

$$[a,b]_1=\left[0,-\frac{b_1}{b_2}\right] \qquad \text{or} \qquad [a,b]=\left[-\frac{b_1}{b_2},0\right], \tag{2.61}$$

$$\psi_k^0(x)=Cx^{k(\gamma-1)+\frac{\gamma}{2}+\log_q\sqrt{1+(1-q^\gamma)\frac{c}{b_1}}}\sqrt{\frac{\left(-\frac{b_2}{b_1}x;q\right)_\infty}{\left(-\frac{b_2+(1-q^\gamma)[\gamma]q^{-\gamma}(a_0-a_1)}{b_1+(1-q^\gamma)c}qx;q\right)_\infty}}, \tag{2.62}$$

$${}^1\psi_k^0(x)=Cx^{k(\gamma-1)+\frac{\gamma}{2}-\frac{\gamma c}{2b_1}}\left(x+\frac{b_1}{b_2}\right)^{-\frac{\gamma^2(a_0-a_1)}{2b_2}+\frac{\gamma c}{2b_1}}. \tag{2.63}$$

(v) If $b_2\neq 0$, $b_1\neq 0$, $b_0=0$, $b_2+(1-q^\gamma)[\gamma]q^{-\gamma}(a_0-a_1)=0$ and $b_1+(1-q^\gamma)c\neq 0$, then

$$\varrho_k(x)=\frac{x^{k(1-2\gamma)-\gamma}}{\left(-\frac{b_2}{b_1}x;q\right)_{k+1}}, \tag{2.64}$$

$$\varrho_k^1(x)=x^{k(1-2\gamma)-\gamma}, \tag{2.65}$$

$$[a,b]_q=\left[0,-q^{-k}\frac{b_1}{b_2}\right]_q \quad \text{for} \quad \frac{b_1}{b_2}<0 \quad \text{or}$$
$$[a,b]_q=\left[-q^{-k}\frac{b_1}{b_2},0\right]_q \quad \text{for} \quad \frac{b_1}{b_2}>0, \tag{2.66}$$

$$[a,b]_1=[0,\infty] \qquad \text{or} \qquad [a,b]=[-\infty,0], \tag{2.67}$$

$$\psi_k^0(x)=Cx^{k(\gamma-1)+\frac{\gamma}{2}+\log_q\sqrt{1+(1-q^\gamma)\frac{c}{b_1}}}\sqrt{\left(-\frac{b_2}{b_1}x;q\right)_\infty}, \tag{2.68}$$

$${}^1\psi_k^0(x)=x^{k(\gamma-1)+\frac{\gamma}{2}-\frac{c\gamma}{2b_1}}\exp\left(-\frac{\gamma^2(a_0-a_1)}{2b_1}x\right). \tag{2.69}$$

(vi) If $b_2=0$, $b_1\neq 0$ and $b_0\neq 0$, then

$$\varrho_k(x)=\frac{x^{(1-\gamma)(2k+1)}}{\left(-\frac{b_1}{b_0}x;q\right)_{k+1}}, \tag{2.70}$$

$$\varrho_k^1(x)=\frac{x^{(1-\gamma)(2k+1)}}{(x+\frac{b_0}{b_1})^{k+1}}, \tag{2.71}$$

$$[a,b]_q=\left[-q^{-k}\frac{b_0}{b_1},\infty\right]_q, \tag{2.72}$$

$$[a,b]_1=\left[-\frac{b_0}{b_1},\infty\right], \tag{2.73}$$

$$\psi_k^0(x)=Cx^{(\gamma-1)(k+\frac{1}{2})}\sqrt{\frac{\left(-\frac{b_1}{b_0}x;q\right)_\infty}{\left(\frac{qx}{y_1};q\right)_\infty\left(\frac{qx}{y_2};q\right)_\infty}}, \tag{2.74}$$

$${}^1\psi_k^0(x)=Cx^{(\gamma-1)(k+\frac{1}{2})}\left(x+\frac{b_0}{b_1}\right)^{-\frac{\gamma c}{2b_1}+\frac{\gamma^2b_0(a_0-a_1)}{2b_1^2}+\frac{1}{2}}\exp\left(-\frac{\gamma^2(a_0-a_1)}{2b_1}x\right). \tag{2.75}$$

(vii) If $b_2 = b_0 = 0$, $b_1 \neq 0$, $a_0 \neq a_1$ and $b_1 + (1-q^\gamma)c \neq 0$, then

$$\varrho_k(x) = x^{k(1-2\gamma)-\gamma}, \tag{2.76}$$

$$\varrho_k^1(x) = x^{k(1-2\gamma)-\gamma}, \tag{2.77}$$

$$[a,b]_q = [0,\infty]_q, \tag{2.78}$$

$$[a,b]_1 = [0,\infty], \tag{2.79}$$

$$\psi_k^0(x) = C\frac{x^{k(\gamma-1)+\frac{\gamma}{2}+\log_q\sqrt{1+(1-q^\gamma)\frac{c}{b_1}}}}{\sqrt{\left(-\frac{(1-q^\gamma)[\gamma]q^{-\gamma}(a_0-a_1)}{b_1+(1-q^\gamma)c}qx;q\right)_\infty}}, \tag{2.80}$$

$${}^1\psi_k^0(x) = Cx^{k(\gamma-1)+\frac{\gamma}{2}-\frac{\gamma c}{2b_1}}\exp\left(-\frac{\gamma^2(a_0-a_1)}{2b_1}x\right). \tag{2.81}$$

(viii) If $b_2 = b_0 = 0$, $b_1 \neq 0$, $a_0 \neq a_1$ and $b_1 + (1-q^\gamma)c = 0$, then

$$\varrho_k(x) = x^{k(1-2\gamma)-\gamma}, \tag{2.82}$$

$$[a,b]_q = [0,\infty]_q, \tag{2.83}$$

$$\psi_k^0(x) = C\frac{x^{k(\gamma-1)+\frac{\gamma+1}{2}+\log_q\sqrt{(1-q^\gamma)[\gamma]q^{-\gamma}\frac{(a_0-a_1)}{b_1}}}}{\sqrt{(-x;q)_\infty(-qx^{-1};q)_\infty}}. \tag{2.84}$$

In the limit $q \to 1$ this case is divergent.

(ix) If $b_2 = b_1 = 0$ and $b_0 \neq 0$, then

$$\varrho_k(x) = x^{(1-\gamma)(2k+1)}, \tag{2.85}$$

$$\varrho_k^1(x) = x^{(1-\gamma)(2k+1)}, \tag{2.86}$$

$$[a,b]_q = [-\infty,\infty]_q, \tag{2.87}$$

$$[a,b]_1 = [-\infty,\infty], \tag{2.88}$$

$$\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}\sqrt{\frac{1}{\left(\frac{qx}{y_1};q\right)_\infty\left(\frac{qx}{y_2};q\right)_\infty}}, \tag{2.89}$$

$${}^1\psi_k^0(x) = Cx^{(\gamma-1)(k+\frac{1}{2})}\exp\left(-\frac{\gamma^2(a_0-a_1)}{4b_0}x^2 - \frac{\gamma c}{2b_0}x\right). \tag{2.90}$$

In all the subcases presented above except of (iii) and (viii), taking the limit $q \to 1$ we have assumed that b_1 does not depend on the parameter q. Similarly, in the all subcases except of (ii), (iii), (v) we have assumed that b_2 does not depend on the parameter q.

Let us now determine the values of the parameters $b_2, b_1, b_0, a_0, a_1, c$ for which the scalar product (2.24) is positively defined

$$\int_a^b |P_k(x)|^2\left|\psi_k^0(x)\right|^2\varrho_k(x)\,d_qx > 0. \tag{2.91}$$

In order to avoid of the cumbersome consideration we will restrict our attention to the generic case (i), when $b_2 \neq 0$ and $b_0 \neq 0$. In this case, we obtain from (2.40) and (2.44) that the positivity condition (2.91) for the scalar product is equivalent to

$$x_1 < 0 < x_2 \tag{2.92}$$

and

$$\begin{cases} \left(\frac{q^{n-k+1}x_1}{y_1};q\right)_\infty \left(\frac{q^{n-k+1}x_1}{y_2};q\right)_\infty > 0 \\ \left(\frac{q^{n-k+1}x_2}{y_1};q\right)_\infty \left(\frac{q^{n-k+1}x_2}{y_2};q\right)_\infty > 0 \end{cases} \tag{2.93}$$

for $n, k \in \mathbb{N} \cup \{0\}$. These conditions are fulfilled if either $y_1 = y_2$ or one of the following inequalities

(a) $y_1 > qx_2$ and $y_2 > qx_2$,
(b) $y_1 < qx_1$ and $y_2 < qx_1$,
(c) $y_1 < qx_1$ and $y_2 > qx_2$

are satisfied.

3. The solution of the eigenvalue problem

The factorization method is known, see [12, 14], to be based on the following relation:

$$\mathbf{A}_k^*\mathbf{A}_k + a_k = Q^{-1}\mathbf{A}_{k+1}\mathbf{A}_{k+1}^* Q + a_{k+1}, \qquad k \in \mathbb{N} \cup \{0\}, \tag{3.1}$$

which gives

$$\mathbf{H}_{k+1}(\mathbf{A}_{k+1}^* Q\psi_k) = \lambda_k(\mathbf{A}_{k+1}^* Q\psi_k), \tag{3.2}$$

provided that $\psi_k \in \mathcal{V}_k$ is a solution of the eigenvalue equation

$$\mathbf{H}_k\psi_k = \lambda_k\psi_k. \tag{3.3}$$

Particularly, if the function $\psi_k^0 \in \mathcal{V}_k$ satisfies (2.23) then

$$\mathbf{H}_k\psi_k^0 = a_k\psi_k^0 \tag{3.4}$$

and

$$\mathbf{H}_k\big(\mathbf{A}_k^* Q \cdots \mathbf{A}_{k+1-n}^* Q\psi_{k-n}^0\big) = a_{k-n}\mathbf{A}_k^* Q \cdots \mathbf{A}_{k+1-n}^* Q\psi_{k-n}^0 \qquad n = 1, \ldots, k. \tag{3.5}$$

One can show (see [6]) that the factorization relations (3.1) are fulfilled for $\mathbf{A}_k$ and $\mathbf{A}_k^*$ given by (2.16), (2.17), (2.18), (2.19), (2.20), (2.21) and (2.22) if

$$a_k = -q^{1-k}\left(a_0[k-1] - a_1[k] + q^\gamma b_2\frac{[k-1][k]}{[\gamma]^2}\right). \tag{3.6}$$

Keeping in mind the factorizing property (3.1) we look for the solutions

$$\psi_k^n(x) = P_k^n(x)\psi_k^0(x) \tag{3.7}$$

of the eigenvalue equation (3.3) under the condition

$$\lambda_k^n = a_{k-n}, \tag{3.8}$$

where a_k is given by (3.6). Substituting (3.7) into (3.3) and using (2.23) one obtains the second-order q-difference equation

$$\begin{aligned} &-q^{-1}D(qx)P_k^n(qx) - B(q^kx)P_k^n(q^{-1}x) + (q^{-1}D(qx) + B(q^kx))P_k^n(x) \\ &\qquad = (1-q^n)(qb_2 + (1-q^\gamma)[\gamma]q^{-\gamma+1}(a_0 - a_1) - q^{2k-n}b_2)x^2 P_k^n(x). \end{aligned} \tag{3.9}$$

for the function $P_k^n(x)$. Equation (3.9) is Hahn equation [11] solutions to which could be expressed in terms of the basic hypergeometric series

$${}_3\phi_2\left(\begin{matrix} a_1, a_2, a_3 \\ d_1, d_2 \end{matrix}\middle| q; x\right) := \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k(a_3;q)_k}{(d_1;q)_k(d_2;q)_k(q;q)_k}x^k, \tag{3.10}$$

see also [13].

In the limit $q \to 1$, equation (3.9) for the function ${}^1P_k^n(x)$ becomes the generalized equation of hypergeometric type (see [15])

$$B^1(x)\frac{d^2}{dx^2}{}^1P_k^n(x) - \left((\gamma^2 a_0 - \gamma^2 a_1 + 2b_2^1(k-1))x + \gamma c + b_1^1(k-1)\right)\frac{d}{dx}{}^1P_k^n(x) + (n\gamma^2(a_0 - a_1) + b_2 n(2k - n - 1))\,{}^1P_k^n(x) = 0 \tag{3.11}$$

and the factorizing condition (3.6) has the form

$$a_k^1 = -a_0(k-1) + a_1 k - b_2^1 \gamma^{-2} k(k-1). \tag{3.12}$$

We show later that after appropriately chosen change of the variable the eigenvalue problem for the operators (2.29) reduces to the wide class of the stationary Schrödinger equations. So, it makes sense to consider the family (2.15) as q-deformation of this class of Schrödinger operators. Thus, solving of (3.3) has a physical motivation too.

We look for the solutions of the eigenvalue problem for the operators (2.29) in the form

$${}^1\psi_k^n(x) = \lim_{q\to 1}\psi_k^n(x) = {}^1P_k^n(x)\,{}^1\psi_k^0(x), \tag{3.13}$$

where ${}^1P_k^n$ are solution (3.11) and ${}^1\psi_k^0$ is given by (2.37). Making the transformation ${}^1\psi_k^n \to \varphi_k^n$ defined by

$${}^1\psi_k^n(x) = \left(x^{2(\gamma^{-1}-1)}B^1(x)\right)^{\frac{2k+1}{4}}\varphi_k^n(z), \tag{3.14}$$

$$dz = \gamma\frac{dx}{(B^1(x))^{\frac{1}{2}}}, \tag{3.15}$$

one maps the family of solutions ${}^1\psi_k^n$ to the family of solutions φ_k^n of Schrödinger equation with the suitable potentials V_k.

Summing up we shall present the solutions of (3.3) in consistency with the list of the weight functions $\varrho_k(x)$ and solutions $\psi_k^0(x)$ of (2.23) given in the previous section.

(i) *Big q-Jacobi orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-2k+n+1}\frac{x_1x_2}{y_1y_2}, \frac{q}{y_1}x \\ q^{-k+1}\frac{x_2}{y_1}, q^{-k+1}\frac{x_1}{y_1}\end{matrix}\middle| q; q\right) \tag{3.16}$$

and the eigenvalues are

$$\lambda_k^n = a_0[n-k+1] - qa_1[n-k] + \frac{q^\gamma b_2}{[\gamma]^2}[n-k+1][k-n]. \tag{3.17}$$

In the limit $q \to 1$, equation (3.11) after the change of the variable $y = 2\frac{x-x_1}{x_2-x_1} - 1$ can be transformed into the equation for the Jacobi orthogonal polynomials and its solution is given by

$${}^1P_k^n(x) = P_n^{(\alpha_k,\beta_k)}(y) = \frac{(\alpha_k+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha_k+\beta_k+1\\ \alpha_k+1\end{matrix}\middle|\frac{1-y}{2}\right), \tag{3.18}$$

where

$$\alpha_k = -\frac{\gamma^2(a_0-a_1)}{b_2} - k - \frac{\gamma(\gamma(a_0-a_1)x_1 + c)}{b_2(x_2-x_1)}, \tag{3.19}$$

$$\beta_k = -1 + \frac{\gamma(\gamma(a_0-a_1)x_1+c)}{b_2(x_2-x_1)}. \tag{3.20}$$

After the transformation (3.15) given by

$$x = \sqrt{\frac{\Delta}{4b_2^2}}\cosh(\gamma^{-1}\sqrt{b_2}(z-c)) - \frac{b_1}{2b_2} \tag{3.21}$$

or

$$x = \sqrt{\frac{\Delta}{4b_2^2}}\sin\gamma^{-1}\sqrt{|b_2|}z - \frac{b_1}{2b_2}, \tag{3.22}$$

we obtain the Schrödinger equation with the *Rosen–Morse II potential* (for $b_2 > 0$)

$$\begin{aligned} V_k(z) = {} & D_1 \coth\gamma^{-1}\sqrt{b_2}(z-c)\,\mathrm{cosech}^2\,\gamma^{-1}\sqrt{b_2}(z-c) \\ & + D_2\,\mathrm{cosech}^2\,\gamma^{-1}\sqrt{b_2}(z-c) + D_3 \end{aligned} \tag{3.23}$$

or the *Eckart II potential* (for $b_2 < 0$)

$$V_k(z) = D_1 \tan\gamma^{-1}\sqrt{|b_2|}z\,\mathrm{sech}\,\gamma^{-1}\sqrt{|b_2|}z + D_2\,\mathrm{sech}^2\,\gamma^{-1}\sqrt{|b_2|}z + D_3, \tag{3.24}$$

respectively, where D_1 and D_2 depend on the parameters $b_2, b_1, b_0, a_0, a_1, c, \gamma$.

(ii) *Big q-Laguerre orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, 0, -\left(\frac{b_1}{b_0} + (1-q^\gamma)\frac{c}{b_0}\right)qx \\ -\left(\frac{b_1}{b_0} + (1-q^\gamma)\frac{c}{b_0}\right)q^{1-k}x_1, -\left(\frac{b_1}{b_0} + (1-q^\gamma)\frac{c}{b_0}\right)q^{1-k}x_2 \end{matrix}\right| q; q\right) \tag{3.25}$$

and the eigenvalues are

$$\lambda_k^n = q^{k-n}(a_0[n-k+1] - a_1[n-k]). \tag{3.26}$$

In the limit $q \to 1$, equation (3.11) after the change of the variable $y = \frac{\gamma^2(a_0-a_1)}{b_1} \times \left(x + \frac{b_0}{b_1}\right)$ can be transformed into the equation for Laguerre's orthogonal polynomials and its solution is given by

$${}^1P_k^n(x) = L_n^{(\alpha_k)}(y) = \frac{(\alpha_k+1)_n}{n!}\,{}_1F_1\left(\left.\begin{matrix} -n \\ \alpha_k+1 \end{matrix}\right| y\right), \tag{3.27}$$

where

$$\alpha_k = \frac{\gamma^2(a_0-a_1)b_0}{b_1^2} - \gamma c + k. \tag{3.28}$$

After the transformation (3.15) given by

$$x = \frac{\gamma^{-2}b_1}{4}z^2 - \frac{b_0}{b_1}, \tag{3.29}$$

we obtain the Schrödinger equation with the *three-dimensional isotropic harmonic oscillator potential*

$$V_k(z) = D_1 z^2 + \frac{D_2}{z^2} + D_3, \tag{3.30}$$

where D_1 and D_2 depend on the parameters $b_1, b_0, a_0, a_1, c, \gamma$.

(iii) *Al-Salam–Carlitz II orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = \left(-\frac{x_1}{x_2}\right)^n q^{\binom{n}{2}}\,{}_2\phi_0\left(\left.\begin{matrix} q^{-n}, \left(\frac{q^k}{x_2}\right)^{-1} \\ 0 \end{matrix}\right| q; \frac{q^{k+1}}{x_1}x\right) \tag{3.31}$$

and the eigenvalues are

$$\lambda_k^n = q^{k-n}(a_0[n-k+1] - a_1[n-k]). \tag{3.32}$$

In the limit $q \to 1$, equation (3.11) after the change of the variable $y = \pm\sqrt{\frac{\gamma^2(a_0-a_1)}{2b_0}} \times \left(x + \frac{c}{\gamma(a_0-a_1)}\right)$ can be transformed into the equation for Hermite orthogonal polynomials and its solution is given by

$$ {}^1P_k^n(x) = H_n(y) = (2y)^n \, {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix}\middle| -\frac{1}{y^2}\right). \tag{3.33} $$

After the transformation (3.15) given by

$$ x = \gamma^{-1}\sqrt{b_0}z, \tag{3.34} $$

we obtain the Schrödinger equation with the *harmonic oscillator potential*

$$ V_k(z) = D_1 z^2 + D_2, \tag{3.35} $$

where D_1 and D_2 depend on the parameters b_0, a_0, a_1, c, γ.

(iv) *Little q-Jacobi orthogonal polynomials*. In this case, solutions of (3.9) are

$$ P_k^n(x) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1}a_k b_k \\ q d_k \end{matrix}\middle| q; -q^{k+1}\frac{b_2}{b_1}x\right), \tag{3.36} $$

where $d_k = q^{-k}(1 + (1 - q^\gamma)\frac{c}{b_1}$, $b_k = q^{-k}\frac{b_1}{b_2}\frac{b_2+(1-q^\gamma)[\gamma]q^{-\gamma}(a_0-a_1)}{b_1+(1-q^\gamma)c}$ and the eigenvalues are

$$ \lambda_k^n = a_0[n-k+1] - qa_1[n-k] + \frac{q^\gamma b_2}{[\gamma]^2}[n-k+1][k-n]. \tag{3.37} $$

In the limit $q \to 1$, equation (3.11) after the change of the variable $y = 2\frac{b_2}{b_1}x + 1$ can be transformed into the equation for the Jacobi orthogonal polynomials and its solution is given by ${}^1P_k^n(x) = P_n^{(\alpha_k,\beta_k)}(y)$, where

$$ \alpha_k = -\frac{\gamma^2(a_0-a_1)}{b_2} + \frac{\gamma c}{b_1} - k \tag{3.38} $$

$$ \beta_k = -k - \frac{\gamma c}{b_1}. \tag{3.39} $$

After the transformation (3.15) given by

$$ x = \frac{b_1}{b_2}\sinh^2\frac{\gamma^{-1}\sqrt{b_2}}{2}(z-c) \tag{3.40} $$

or

$$ x = \frac{b_1}{b_2}\sin^2\frac{\gamma^{-1}\sqrt{|b_2|}}{2}z, \tag{3.41} $$

we obtain the Schrödinger equation with the *Pöschl–Teller II potential* (for $b_2 > 0$)

$$ V_k(z) = D_1\,\mathrm{cosech}^2\frac{\gamma^{-1}\sqrt{b_2}}{2}(z-c) + D_2\,\mathrm{sech}^2\frac{\gamma^{-1}\sqrt{b_2}}{2}(z-c) + D_3 \tag{3.42} $$

or the *Pöschl–Teller I potential* (for $b_2 < 0$)

$$ V_k(z) = D_1\,\mathrm{cosec}^2\frac{\gamma^{-1}\sqrt{|b_2|}}{2}z + D_2\sec^2\frac{\gamma^{-1}\sqrt{|b_2|}}{2}z + D_3, \tag{3.43} $$

respectively, where D_1 and D_2 depend on the parameters $b_2, b_1, a_0, a_1, c, \gamma$.

(v) *Little q-Laguerre/Wall orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ q^{-k+1}(1+(1-q^\gamma)\frac{c}{b_1} \end{matrix}\middle| q; -q^{k+1}\frac{b_2}{b_1}x\right) \tag{3.44}$$

and the eigenvalues are

$$\lambda_k^n = q^{k-n}(a_0[n-k+1] - a_1[n-k]). \tag{3.45}$$

In the limit $q \to 1$, solution (3.44) gives (3.27). Taking the coordinate transformation given by (3.14), (3.15) one obtains the case of *three-dimensional isotropic harmonic oscillator potential*.

(vi) *q-Meixner orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = {}_2\phi_1\left(\begin{matrix} q^{-n}, \frac{qx}{y_1} \\ -\frac{q^{1-k}b_0}{y_1 b_1} \end{matrix}\middle| q; -q^{n-k+1}\frac{b_0}{y_2 b_1}\right) \tag{3.46}$$

and the eigenvalues are

$$\lambda_k^n = a_0[n-k+1] - q a_1[n-k]. \tag{3.47}$$

In the limit $q \to 1$, solution (3.46) gives (3.27). Taking the coordinate transformation given by (3.14), (3.15) one obtains the case of *three-dimensional isotropic harmonic oscillator potential*.

(vii) *q-Laguerre orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = \frac{1}{(q;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n}, -\frac{(1-q^\gamma)[\gamma]q^{-\gamma}(a_0-a_1)}{b_1+(1-q^\gamma)c}x \\ 0 \end{matrix}\middle| q; -q^{n-k+1}\left(1+(1-q^\gamma)\frac{c}{b_1}\right)\right) \tag{3.48}$$

and the eigenvalues are

$$\lambda_k^n = a_0[n-k+1] - q a_1[n-k]. \tag{3.49}$$

In the limit $q \to 1$, solution (3.48) gives (3.27) with $b_0 = 0$. Taking the coordinate transformation given by (3.14), (3.15) one obtains the case of *three-dimensional isotropic harmonic oscillator potential*.

(viii) *Stieltjes–Wigert orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = \frac{1}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}\middle| q; -q^{n-k+1}(1-q^\gamma)[\gamma]q^{-\gamma}\frac{a_0-a_1}{b_1}\right) \tag{3.50}$$

and the eigenvalues are

$$\lambda_k^n = a_0[n-k+1] - q a_1[n-k]. \tag{3.51}$$

The limit $q \to 1$ case does not exist.

(ix) *Al-Salam–Carlitz II orthogonal polynomials*. In this case, solutions of (3.9) are

$$P_k^n(x) = \left(-\frac{y_2}{y_1}\right)^n q^{-\binom{n}{2}}\,{}_2\phi_0\left(\begin{matrix} q^{-n}, \frac{qx}{y_1} \\ - \end{matrix}\middle| q; -q^n\frac{y_1}{y_2}\right) \tag{3.52}$$

and the eigenvalues are

$$\lambda_k^n = a_0[n-k+1] - q a_1[n-k]. \tag{3.53}$$

In the limit $q \to 1$, solution (3.52) gives (3.33). Taking the coordinate transformation given by (3.14), (3.15) one obtains the case of *the harmonic oscillator potential*.

In the papers [1–5], one can find the solutions for different models of the q-deformed harmonic oscillator.

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